Restricted Sum Formula of Multiple Zeta Values

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1 Introduction

For fixed positive integer d and d-tuple of positive integers $(s_1, \dots, s_d)$ with $s_1 > 1$, the multiple zeta value $\zeta(s_1, \dots, s_d)$ is defined by

$$\zeta(s_1, \dots, s_d) = \sum_{k_1 > \cdots > k_d > 0} k_1^{-s_1} \cdots k_d^{-s_d}, \tag{1}$$

where d is called the *depth* and $s_1 + \cdots + s_d$ the *weight*. The double zeta values were studied by Euler [1] who derived many identities such as follows:

$$\sum_{k=2}^{2n-1} (-1)^k \zeta(k, 2n-k) = \frac{1}{2}\zeta(2n),$$

$$\sum_{k=2}^{2n-1} \zeta(k, 2n-k) = \zeta(2n),$$

from which we can easily get (see [2, Theorem 1])

$$\sum_{k=1}^{n-1} \zeta(2k, 2n-2k) = \frac{3}{4}\zeta(2n). \tag{2}$$

Using the stuffle relation $\zeta(2k)\zeta(2n-2k) = \zeta(2k, 2n-2k) + \zeta(2n-2k, 2k) + \zeta(2n)$ we see immediately

$$\sum_{k=1}^{n-1} \zeta(2k)\zeta(2n-2k) = \frac{2n+1}{2}\zeta(2n). \tag{3}$$

Recently, Hoffman [3] extended (2) to arbitrary depths. Moreover, similar formulas have been obtained for some special type Hurwitz-zeta values [4] and alternating Euler sums [5]. In this paper we consider the following restricted sum of multiple zeta values

$$Q(4n, d) = \sum_{\substack{j_1 + \cdots + j_d = n \\ j_1, \dots, j_d > 0}} \zeta(4j_1, \dots, 4j_d).$$

Our main theorem is

Theorem 1.1. *For any positive integers $n \geq d \geq 3$,*

$$Q(4n,d) = \sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{j=0}^{2k+1} \frac{2^{k+2}(-1)^{\lfloor \frac{k}{2} \rfloor + j + d}}{(2k+1)!} \binom{2k+1}{j} \binom{\frac{j-2}{4}}{d} \zeta(4n-2k)\pi^{2k}$$
$$+ \sum_{k=0}^{\lfloor \frac{d-2}{4} \rfloor} \sum_{j=0}^{4k+2} \frac{2^{2k+5}(-1)^{k+j+d}}{(4k+2)!} \binom{4k+2}{j} \binom{\frac{j-2}{4}}{d} \left(Q(4n-4k,2) - \frac{7}{8}\zeta(4n-4k) \right) \pi^{4k}.$$

Remark 1.2. For $d = 2$, it's easy to prove by stuffle relation that

$$Q(4n,2) = \frac{1}{2} \sum_{k=1}^{n-1} \zeta(4k)\zeta(4n-4k) - \frac{n-1}{2}\zeta(4n)$$

for $n \geq 2$. However, it is an intriguing problem to find a compact formula similar to (3).

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2 The generating function of $Q(4n,d)$

Recall that the symmetric function of the infinitely many variables $x_1, x_2, \cdots$ form a subring Sym of $\mathbb{Q}[x_1, x_2, \cdots]$ which is invariant under all the permutations of the variables. Let $e_j = \sum_{k_1 < \cdots < k_j} x_{k_1} \ldots x_{k_j}$ be the j-th elementary function. Following Hoffman [3] let's consider its generating function

$$E(t) = \prod_{j=1}^{\infty} (1 + tx_j) = \sum_{j=0}^{\infty} e_j t^j$$

and define $\varepsilon : \text{Sym} \to \mathbb{R}$ to be the evaluation map such that $\varepsilon(x_j) = \dfrac{1}{j^4}$. Let

$$F(s,t) = \prod_{j=1}^{\infty} (1 + tsx_j + ts^2x_j^2 + \cdots).$$

Then it is not hard to see that the generating function of $Q(4n,d)$ is given by

$$\varepsilon\big(F(s,t)\big) = \sum_{n=0}^{\infty} Q(4n,d)t^d s^n.$$

First we need the following lemma.

Lemma 2.1. *We have*

$$\varepsilon(F(s,t)) = \frac{\sin\pi\sqrt[4]{s(1-t)}\cdot\sinh\pi\sqrt[4]{s(1-t)}}{\sqrt{1-t}\sin\pi\sqrt[4]{s}\cdot\sinh\pi\sqrt[4]{s}}.$$

Proof. We have

$$\begin{aligned}\prod_{j=1}^{\infty}(1+tsx_j+ts^2x_j^2+\cdots) &= \prod_{j=1}^{\infty}\left(1+t\frac{sx_j}{1-sx_j}\right)\\ &= \frac{\prod_{j=1}^{\infty}(1-s(1-t)x_j)}{\prod_{j=1}^{\infty}(1-sx_j)} = \frac{E(-s(1-t))}{E(-s)}.\end{aligned}$$

Further,

$$\varepsilon(E(-t)) = \prod_{i=1}^{\infty}\left(1-\frac{t}{i^4}\right) = \prod_{i=1}^{\infty}\left(1-\frac{\sqrt{t}}{i^2}\right)\left(1+\frac{\sqrt{t}}{i^2}\right) = \frac{\sin\pi\sqrt[4]{t}\cdot\sinh\pi\sqrt[4]{t}}{\pi^2\sqrt{t}}.$$

The lemma follows immediately. □

Let $f(x) = \sin x\cdot\sinh x/(2x^2)$. The following lemma provides its series expansion.

Lemma 2.2. *We have*

$$f(x) = \sum_{k=0}^{\infty}\frac{(-1)^k4^k}{(4k+2)!}x^{4k}.$$

Proof. Using the well-known formula $\sin x = (e^{ix}-e^{-ix})/(2i)$ we obtain

$$\begin{aligned}f(x) &= \frac{1}{2}\cdot\frac{e^{ix}-e^{-ix}}{2ix}\cdot\frac{e^{x}-e^{-x}}{2x}\\ &= \frac{e^{(i+1)x}+e^{-(i+1)x}-(e^{(i-1)x}+e^{-(i-1)x})}{8ix^2}\\ &= \frac{1}{4ix^2}\left(\sum_{n=0}^{\infty}\frac{(2i)^nx^{2n}}{(2n)!}-\sum_{n=0}^{\infty}\frac{(-2i)^nx^{2n}}{(2n)!}\right)\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k4^k}{(4k+2)!}x^{4k},\end{aligned}$$

as desired. □

3 Proof of Theorem 1.1

Let $g(t) = f(\sqrt[4]{t})$. Then

$$\frac{g(s(1-t))}{g(s)} = \varepsilon(F(s/\pi^4,t)) = \frac{1}{g(s)}\sum_{k=0}^{\infty}\frac{(-1)^k4^k}{(4k+2)!}s^k(1-t)^k.$$

Write

$$\frac{g(s(1-t))}{g(s)} = \sum_{d=0}^{\infty} G_d(s) t^d.$$

By the above expression, we have

$$G_d(s) = \frac{(-s)^d}{g(s)d!} D^d g(s),$$

where D^d denotes the d-th derivative with respect to s. Set

$$G_d(s) = X_d(s)\sqrt[4]{s}\cot\sqrt[4]{s} + Y_d(s)\sqrt[4]{s}\coth\sqrt[4]{s} + Z_d(s)\cot\sqrt[4]{s}\coth\sqrt[4]{s} + W_d(s) \tag{4}$$

which yields easily

$$\begin{aligned}\frac{(-1)^s D^d g(s)}{d!} &= X_d(s) s^{-d-\frac{1}{4}} \cos s^{\frac{1}{4}} \sinh s^{\frac{1}{4}} + Y_d(s) s^{-d-\frac{1}{4}} \sin s^{\frac{1}{4}} \cosh s^{\frac{1}{4}} \\ &+ Z_d(s) s^{-d-\frac{1}{2}} \cos s^{\frac{1}{4}} \cosh s^{\frac{1}{4}} + W_d(s) s^{-d-\frac{1}{2}} \sin s^{\frac{1}{4}} \sinh s^{\frac{1}{4}}.\end{aligned}$$

To determine the coefficients $X_d(s), Y_d(s), Z_d(s)$ and $W_d(s)$ we differentiate the both sides of the above equation to get the following system of recursive differential equations

$$\begin{cases} (d+1)X_{d+1}(s) = -sX_d'(s) + \left(d + \frac{1}{4}\right)X_d(s) - \frac{1}{4}Z_d(s) - \frac{1}{4}W_d(s), \\ (d+1)Y_{d+1}(s) = -sY_d'(s) + \left(d + \frac{1}{4}\right)Y_d(s) + \frac{1}{4}Z_d(s) - \frac{1}{4}W_d(s), \\ (d+1)Z_{d+1}(s) = -\frac{\sqrt{s}}{4}X_d(s) - \frac{\sqrt{s}}{4}Y_d(s) - sZ_d'(s) + \left(d + \frac{1}{2}\right)Z_d(s), \\ (d+1)W_{d+1}(s) = \frac{\sqrt{s}}{4}X_d(s) - \frac{\sqrt{s}}{4}Y_d(s) + \left(d + \frac{1}{2}\right)W_d(s) - sW_d'(s), \end{cases}$$

with the initial conditions $X_0(s) = Y_0(s) = Z_0(s) = 0$ and $W_0(s) = 1$. Let $x_d(u) = X_d(u^2), y_d(u) = Y_d(u^2), z_d(u) = Z_d(u^2)$ and $w_d(u) = W_d(u^2)$. The above system is changed into the following system:

$$\begin{cases} (d+1)x_{d+1}(u) = -\frac{u}{2}x_d'(u) + \left(d + \frac{1}{4}\right)x_d(u) - \frac{1}{4}z_d(u) - \frac{1}{4}w_d(u), \\ (d+1)y_{d+1}(u) = -\frac{u}{2}y_d'(u) + \left(d + \frac{1}{4}\right)y_d(u) + \frac{1}{4}z_d(u) - \frac{1}{4}w_d(u), \\ (d+1)z_{d+1}(u) = -\frac{u}{4}x_d(u) - \frac{u}{4}y_d(u) - \frac{u}{2}z_d'(u) + \left(d + \frac{1}{2}\right)z_d(u), \\ (d+1)w_{d+1}(u) = \frac{u}{4}x_d(u) - \frac{u}{4}y_d(u) + \left(d + \frac{1}{2}\right)w_d(u) - \frac{u}{2}w_d'(u). \end{cases} \tag{5}$$

Define

$$\begin{cases} \alpha(u,v) = \sum_{d\geq 0} x_d(u)v^d = \sum_{d\geq 0} \tilde{x}_d(v)u^d, \\ \beta(u,v) = \sum_{d\geq 0} y_d(u)v^d = \sum_{d\geq 0} \tilde{y}_d(v)u^d, \\ \gamma(u,v) = \sum_{d\geq 0} z_d(u)v^d = \sum_{d\geq 0} \tilde{z}_d(v)u^d, \\ \delta(u,v) = \sum_{d\geq 0} w_d(u)v^d = \sum_{d\geq 0} \tilde{w}_d(v)u^d. \end{cases} \tag{6}$$

Multiplying the system (5) by v^d and then taking the sum $\sum_{d\geq 0}$ we get:

$$\begin{cases} \frac{\partial \alpha}{\partial v} =v\frac{\partial \alpha}{\partial v} + \frac{1}{4}\alpha - \frac{u}{2}\frac{\partial \alpha}{\partial u} - \frac{1}{4}\gamma - \frac{1}{4}\delta, \\ \frac{\partial \beta}{\partial v} =v\frac{\partial \beta}{\partial v} + \frac{1}{4}\beta - \frac{u}{2}\frac{\partial \beta}{\partial u} + \frac{1}{4}\gamma - \frac{1}{4}\delta, \\ \frac{\partial \gamma}{\partial v} =v\frac{\partial \gamma}{\partial v} + \frac{1}{2}\gamma - \frac{u}{2}\frac{\partial \gamma}{\partial u} - \frac{u}{4}\alpha - \frac{u}{4}\beta, \\ \frac{\partial \delta}{\partial v} =v\frac{\partial \delta}{\partial v} + \frac{1}{2}\delta - \frac{u}{2}\frac{\partial \delta}{\partial u} + \frac{u}{4}\alpha - \frac{u}{4}\beta. \end{cases}$$

Comparing the coefficients of u^n we get

$$\begin{cases} \tilde{x}_n'(v) =v\tilde{x}_n'(v) + \frac{1}{4}\tilde{x}_n(v) - \frac{n}{2}\tilde{x}_n(v) - \frac{1}{4}\tilde{z}_n(v) - \frac{1}{4}\tilde{w}_n(v), \\ \tilde{y}_n'(v) =v\tilde{y}_n'(v) + \frac{1}{4}\tilde{y}_n(v) - \frac{n}{2}\tilde{y}_n(v) + \frac{1}{4}\tilde{z}_n(v) - \frac{1}{4}\tilde{w}_n(v), \\ \tilde{z}_n'(v) =v\tilde{z}_n'(v) + \frac{1}{2}\tilde{z}_n(v) - \frac{n}{2}\tilde{z}_n(v) - \frac{1}{4}\tilde{x}_{n-1}(v) - \frac{1}{4}\tilde{y}_{n-1}(v), \\ \tilde{w}_n'(v) =v\tilde{w}_n'(v) + \frac{1}{2}\tilde{w}_n(v) - \frac{n}{2}\tilde{w}_n(v) + \frac{1}{4}\tilde{x}_{n-1}(v) - \frac{1}{4}\tilde{y}_{n-1}(v), \end{cases} \tag{7}$$

By definition (6), we see that the system has the following initial values: $\tilde{x}_n(0) = 0, \tilde{y}_n(0) = 0, \tilde{z}_n(0) = 0$ for all $n \geq 0$ and $\tilde{w}_n(0) = 0$ for all $n \geq 1$. But for $\tilde{w}_0(v)$ we have from (5)

$$w_0(0) = 1, \quad w_d(0) = \frac{2d-1}{2d} w_{d-1}(0) \quad \forall d \geq 1.$$

It follows that $w_d(0) = \binom{2d}{d}/2^{2d}$ which yields easily

$$\tilde{w}_0(v) = \sum_{d\geq 0} w_d(0)v^d = (1-v)^{-\frac{1}{2}}.$$

Similarly we see that $\tilde{z}_0(v) = 0$. Solving (7) recursively starting from the first two equations in (7) we find the following functions are the unique solution satisfying the initial conditions:

$$\begin{cases} \tilde{x}_n(v) = \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\lfloor\frac{n+2}{2}\rfloor+j}}{j!(2n+1-j)!}(1-v)^{\frac{j-2}{4}}; \\ \tilde{y}_n(v) = \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\lfloor\frac{n+3}{2}\rfloor+j}}{j!(2n+1-j)!}(1-v)^{\frac{j-2}{4}}; \\ \tilde{z}_n(v) = (1-(-1)^n)\sum_{j=0}^{2n} \frac{2^{n-1}(-1)^{\frac{n-1}{2}+j}}{j!(2n-j)!}(1-v)^{\frac{j-2}{4}}; \\ \tilde{w}_n(v) = (1+(-1)^n)\sum_{j=0}^{2n} \frac{2^{n-1}(-1)^{\frac{n}{2}+j}}{j!(2n-j)!}(1-v)^{\frac{j-2}{4}}. \end{cases}$$

Using (6) we can solve $x_n(v), y_n(v), z_n(v)$ and $w_n(v)$ and get

$$x_d(u) = \sum_{n=0}^{\lfloor\frac{d-1}{2}\rfloor} \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\lfloor\frac{n+2}{2}\rfloor+j+d}}{(2n+1)!}\binom{2n+1}{j}\binom{\frac{j-2}{4}}{d}u^n;$$

$$y_d(u) = \sum_{n=0}^{\lfloor\frac{d-1}{2}\rfloor} \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\lfloor\frac{n+3}{2}\rfloor+j+d}}{(2n+1)!}\binom{2n+1}{j}\binom{\frac{j-2}{4}}{d}u^n;$$

$$z_d(u) = \sum_{n=0}^{2\lfloor\frac{d-2}{4}\rfloor+1} \sum_{j=0}^{2n}(1-(-1)^n)\frac{2^{n-1}(-1)^{\frac{n-1}{2}+j+d}}{(2n)!}\binom{2n}{j}\binom{\frac{j-2}{4}}{d}u^n;$$

$$w_d(u) = \sum_{n=0}^{2\lfloor\frac{d}{4}\rfloor} \sum_{j=0}^{2n}(1+(-1)^n)\frac{2^{n-1}(-1)^{\frac{n}{2}+j+d}}{(2n)!}\binom{2n}{j}\binom{\frac{j-2}{4}}{d}u^n.$$

Thus

$$X_d(s) = \sum_{n=0}^{\lfloor\frac{d-1}{2}\rfloor} \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\lfloor\frac{n+2}{2}\rfloor+j+d}}{(2n+1)!}\binom{2n+1}{j}\binom{\frac{j-2}{4}}{d}s^{\frac{n}{2}};$$

$$Y_d(s) = \sum_{n=0}^{\lfloor\frac{d-1}{2}\rfloor} \sum_{j=0}^{2n+1} \frac{2^n(-1)^{\lfloor\frac{n+3}{2}\rfloor+j+d}}{(2n+1)!}\binom{2n+1}{j}\binom{\frac{j-2}{4}}{d}s^{\frac{n}{2}};$$

$$Z_d(s) = \sum_{n=0}^{\lfloor\frac{d-2}{4}\rfloor} \sum_{j=0}^{4n+2} \frac{2^{2n+1}(-1)^{n+j+d}}{(4n+2)!}\binom{4n+2}{j}\binom{\frac{j-2}{4}}{d}s^{n+1/2};$$

$$W_d(s) = \sum_{n=0}^{\lfloor\frac{d}{4}\rfloor} \sum_{j=0}^{4n} \frac{2^{2n}(-1)^{n+j+d}}{(4n)!}\binom{4n}{j}\binom{\frac{j-2}{4}}{d}s^{n}.$$

By the well-known formulas

$$z \cot z = -2\sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} z^{2n}, \quad z \coth z = -2\sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n)}{\pi^{2n}} z^{2n},$$

we obtain

$$\sqrt[4]{s} \cot \sqrt[4]{s} = -2\sum_{n=0}^{\infty} \frac{\zeta(2n)}{\pi^{2n}} s^{\frac{n}{2}}, \quad \sqrt[4]{s} \coth \sqrt[4]{s} = -2\sum_{n=0}^{\infty} (-1)^n \frac{\zeta(2n)}{\pi^{2n}} s^{\frac{n}{2}},$$

and

$$\sqrt{s} \cot \sqrt[4]{s} \cdot \coth \sqrt[4]{s} = 4\sum_{k=0}^{\infty} \sum_{m+l=k} (-1)^m \frac{\zeta(2m)\zeta(2l)}{\pi^{2k}} s^{\frac{k}{2}} = 4\sum_{k=0}^{\infty} \sum_{m+l=2k} (-1)^m \frac{\zeta(2m)\zeta(2l)}{\pi^{4k}} s^{k}.$$

Here by exchanging m and l we notice that the inner sum vanishes if k is odd. Hence the coefficient of s^n in $G_d(\pi^4 s)$ is

$$\begin{aligned} Q(4n,d) =& 2\sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{j=0}^{2k+1} \frac{2^k(-1)^{\lfloor \frac{k}{2} \rfloor + j + d}}{(2k+1)!} \binom{2k+1}{j} \binom{\frac{j-2}{4}}{d} \zeta(4n-2k)\pi^{2k} \\ &+ 2\sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{j=0}^{2k+1} (-1)^k \frac{2^k(-1)^{\lfloor \frac{k+1}{2} \rfloor + j + d}}{(2k+1)!} \binom{2k+1}{j} \binom{\frac{j-2}{4}}{d} \zeta(4n-2k)\pi^{2k} \\ &+ 4\sum_{k=0}^{\lfloor \frac{d-2}{4} \rfloor} \sum_{j=0}^{4k+2} \frac{2^{2k+1}(-1)^{k+j+d}}{(4k+2)!} \binom{4k+2}{j} \binom{\frac{j-2}{4}}{d} \left(\sum_{\substack{m,l\geq 0,\\ m+l=2n-2k}} (-1)^m \zeta(2m)\zeta(2l) \right) \pi^{4k} \end{aligned}$$

since $W_d(s)$ has degree less than n. Observe that the first two lines are the same and for any positive integer w

$$\begin{aligned} \sum_{\substack{m,l\geq 0,\\ m+l=2w}} (-1)^m \zeta(2m)\zeta(2l) =& 2\sum_{l=1}^{w-1} \zeta(4l)\zeta(4w-4l) - \sum_{l=1}^{2w-1} \zeta(2l)\zeta(4w-2l) - \zeta(4w) \\ =& 4Q(4w,2) + (2w-3)\zeta(4w) - \frac{4w+1}{2}\zeta(4w) \\ =& 4Q(4w,2) - \frac{7}{2}\zeta(4w) \end{aligned}$$

by stuffle relation $\zeta(4m)\zeta(4l) = \zeta(4m,4l) + \zeta(4l,4m) + \zeta(4m+4l)$ and equation (3). Therefore we finally get

$$\begin{aligned} Q(4n,d) = & 4\sum_{k=0}^{\lfloor \frac{d-1}{2} \rfloor} \sum_{j=0}^{2k+1} \frac{2^k(-1)^{\lfloor \frac{k}{2} \rfloor + j + d}\zeta(4n-2k)\pi^{2k}}{(2k+1)!} \binom{2k+1}{j} \binom{\frac{j-2}{4}}{d} \zeta(4n-2k)\pi^{2k} \\ &+ 4\sum_{k=0}^{\lfloor \frac{d-2}{4} \rfloor} \sum_{j=0}^{4k+2} \frac{2^{2k+1}(-1)^{k+j+d}}{(4k+2)!} \binom{4k+2}{j} \binom{\frac{j-2}{4}}{d} \left(4Q(4n-4k,2) - \frac{7}{2}\zeta(4n-4k) \right) \pi^{4k}. \end{aligned}$$

This concludes the proof of Theorem 1.1 and this paper.

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